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PETTIS INTEGRABILITY

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ABSTRACT. A weakly measurable function $f : \Omega \to X$ is said to be determined by a subspace D of X if for each $x^* \in X^*$, $x^*|_D = 0$ implies that $x^* f = 0$ a.e. For a given Dunford integrable function $f : \Omega \to X$ with a countably additive indefinite integral we show that f is Pettis integrable if and only if f is determined by a weakly compactly generated subspace of X if and only if f is determined by a subspace which has Mazur's property.

We show that if $f : \Omega \to X$ is Pettis integrable then there exists a sequence (φ_n) of X valued simple functions such that for all $x^* \in X^*$, $x^* f = \lim_n x^* \varphi_n$ a.e. if and only if f is determined by a separable subspace of X.

For a bounded weakly measurable function $f : \Omega \to X^*$ into a dual of a weakly compactly generated space, we show that f is Pettis integrable if and only if f is determined by a separable subspace of X^* if and only if f is weakly equivalent to a Pettis integrable function that takes its range in $\operatorname{cor}^*_f(\Omega)$.

1. INTRODUCTION

It is well known [9] that if $(\Omega, \Sigma, \lambda)$ is a finite measure space, X a Banach space with dual X^*, and $f : \Omega \to X$ weakly measurable, then f is Pettis integrable if and only if the operator $T : X^* \to L_1(\lambda)$, $x^* \mapsto x^* f$ is weak*-to-weak continuous. However, unless weak*-to-weak continuity is implied by sequential weak*-to-weak continuity of T, this criterion is very hard to test directly. In [9] R. Huff demonstrates how one can, in certain cases, bypass these difficulties. In this paper we generalize the ideas put forth in [9] and show how far these generalizations go towards characterizing Pettis integrability.

Let us fix some terminology and notation. The dual of a Banach space X will be denoted by X^* and its closed unit ball will be denoted by B_X. Throughout, $(\Omega, \Sigma, \lambda)$ will denote a finite measure space. For convenience we assume the measure space to be complete. A function $f : \Omega \to X$ is *Dunford integrable* provided the composition $Tx^* = x^* f$ is in $L_1(\lambda)$ for every x^* in X^*. In that case, the operator $T : X^* \to L_1(\lambda)$ is bounded (the closed graph theorem). If T^* denotes the adjoint of T then $T^* \chi_E$ is in X^{**} for all E in Σ. The element $T^* \chi_E$ is called the Dunford integral of f over E and is denoted

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by $(D)\text{-}\int_E f\,d\lambda$. The function f is called *Pettis integrable* if and only if its Dunford integral is an element of the natural image of X in X^{**}. In that case we write $(P)\text{-}\int_E f\,d\lambda$ instead of $(D)\text{-}\int_D f\,d\lambda$. The function $\nu : \Sigma \to X^{**}$, $E \mapsto (D)\text{-}\int_E f\,d\lambda$ is called the *indefinite integral* of f and it can be shown to be countably additive if and only if T is weakly compact if and only if $\{x^*f : x^* \in B_{X^*}\}$ is uniformly integrable in $L_1(\lambda)$.

If $f : \Omega \to X^*$ is weak* measurable [5], and $f(\cdot)x$ is in $L_1(\lambda)$ for all x in X, we say that f is *weak* integrable.* In that case, to each E in Σ there corresponds an element x_E^* in X^* such that $x_E^*(x) = \int_E f(\cdot)x\,d\lambda$ for all x in X. The element x_E^* is called the *weak* integral of f over E* and is denoted by $w^*\text{-}\int_E f\,d\lambda$.

If K is a subset of a Banach space X, its linear span will be denoted by $\operatorname{span}(K)$, its convex hull by $\operatorname{co}(K)$, its norm closure by $\operatorname{Cl}(K)$, and its weak* closure by $w^*\text{-}\operatorname{Cl}(K)$.

A Banach space X is said to be *weakly compactly generated* (WCG) if there is a weakly compact subset K of X whose linear span is dense in X. X has *Mazur's property* if the sequentially weak* continuous functionals on X^* are in X.

2. Pettis integrability

Let $f : \Omega \to X$ be a weakly measurable function and assume there is a subspace D of X such that whenever $x^*|_D = 0$ then $x^*f = 0$ almost everywhere. In that case, for each x^* in X^*, there exists a sequence (φ_n) of D-valued simple functions with $x^*f = \lim x^*\varphi_n$ almost everywhere. Indeed, if $x^*|_D = 0$ choose $\varphi_n = 0$ for all n. Otherwise, find a sequence ϕ_n of real-valued simple functions with $\lim \phi_n = x^*f$. Then, choose an element d in D such that $x^*(d) = 1$, and if we let $\varphi_n = d\phi_n$, then (φ_n) is a sequence of D-valued simple functions, and $\lim x^*\varphi_n = x^*f$. This property of the function f is formulated in the following definition.

Definition 2.1. A weakly measurable function $f : \Omega \to X$ is said to be determined by a subspace D of X if one of the following equivalent statements holds.

(a) If x^* restricted to D equals zero then x^*f equals zero a.e.

(b) For each x^* in X^* there exists a sequence (φ_n) of D-valued simple functions such that $x^*f = \lim x^*\varphi_n$ a.e.

All strongly measurable functions (see [5]) are clearly determined by separable spaces. In [9], R. Huff calls such functions *separable-like* and shows that Dunford integrable functions with countably additive indefinite integrals are Pettis integrable whenever they are separable-like. The converse is not true [6].

Let $f : \Omega \to X$ be Dunford integrable and assume f is determined by a subspace D of X. Let $T : X^* \to L_1(\lambda)$, $x^* \mapsto x^*f$. Define an operator

$$T_D : D^* \to L_1(\lambda),$$

by

$$T_D(d^*) = T(d^*_{\text{ext}}),$$

where d^*_{ext} is any extension of d^* to all of X.

Proposition 2.2. *Let $f : \Omega \to X$ be a Dunford integrable function determined by a subspace D of X. The operator T_D defined above is well defined and bounded. Furthermore, T_D is weak*-to-weak continuous if and only if T is weak*-to-weak continuous.*

Proof. That T_D is well defined and bounded is clear. Also, it is clear that T is weak*-to-weak continuous if T_D is weak*-to-weak continuous; so assume T is weak*-to-weak continuous. Let $(d_\alpha^*)_{\alpha\in A}$ be a net in B_{D^*} converging weak* to zero. Choose a net $(x_\alpha^*)_{\alpha\in A}$ in B_{X^*} such that $x_\alpha^*|_D = d_\alpha^*$. If x^* is any weak* cluster point of (x_α^*) then $x^*|_D = 0$. Let h be any weak cluster point of $(Tx_\alpha^*)_{\alpha\in A}$ and let V a weak neighborhood system of h. Let $F = A \times \mathsf{V}$ with $(\alpha, V) \geq (\beta, U)$ meaning $\alpha \geq \beta$ and $V \subseteq U$. Then F is a directed set. Since h is a weak cluster point of the net $(Tx_\alpha^*)_{\alpha\in A}$, the set $T^{-1}(V) \cap \{x_\gamma^* : \gamma \geq \alpha\}$ is nonempty for all (α, V) in F. For each (α, V) in F choose an element $y_{(\alpha,V)}^*$ in $T^{-1}(V) \cap \{x_\gamma^* : \gamma \geq \alpha\}$. Then $(y_{(\alpha,V)}^*)_{(\alpha,V)\in F}$ is a subnet of $(x_\alpha^*)_{\alpha\in A}$ and $Ty_{(\alpha,V)}^* \xrightarrow[(\alpha,V)]{} h$ weakly. Let z^* be any weak* cluster point of $(y_{(\alpha,V)}^*)_{(\alpha,V)\in F}$ and choose a subnet (y_β^*) of $(y_{(\alpha,V)}^*)_{(\alpha,V)\in F}$ that converges weak* to z^*. Then (Ty_β^*) is a subnet of $(Ty_{(\alpha,V)}^*)_{(\alpha,V)\in F}$ and $Ty_\beta^* \xrightarrow[\beta]{} h$ weakly. But T is weak*-to-weak continuous and therefore we also have that $Ty_\beta^* \xrightarrow[\beta]{} Tz^*$ weakly. Thus, $Tz^* = h$. But $z^*|_D = 0$, being a weak* cluster point of $(x_\alpha^*)_{\alpha\in A}$. Hence, $h \in Tz^* = 0$.

We have shown that 0 is the only weak cluster point of $(Tx_\alpha^*)_{\alpha\in A}$ which implies that $Tx_\alpha^* \xrightarrow[\alpha]{} 0$ weakly, and therefore $T_D d_\alpha^* = Tx_\alpha^* \xrightarrow[\alpha]{} 0$ weakly. □

If D is a subspace of X, let $\sigma(X^*, D)$ denote the topology on X^* with basic neighborhoods of zero

$$W(0; d_1, d_2, \ldots, d_n, \varepsilon) = \{x^* \in X^* : |x^*(d_i)| < \varepsilon,\ 1 \leq i \leq n\},$$

where $d_1, d_2, \ldots, d_n$ are in D. $\sigma(X^*, D)$ is the coarsest topology on X^* with respect to which all the elements in D are continuous. The following corollary is basically a reformulation of the above proposition.

Corollary 2.3. *A Dunford integrable function $f : \Omega \to X$ determined by D is Pettis integrable if and only if T is $\sigma(X^*, D)$-to-weak continuous.*

Proof. T is $\sigma(X^*, D)$-to-weak continuous if and only if T_D is weak*-to-weak continuous if and only if T is weak*-to-weak continuous if and only if f is Pettis integrable. □

Theorem 2.4. *Let $f : \Omega \to X$ be a Dunford integrable function determined by a subspace D. If T is weakly compact (resp. norm compact), then T is sequentially $\sigma(X^*, D)$-to-weak (resp. sequentially $\sigma(X^*, D)$-to-norm) continuous.*

Proof. Assume T is compact and let (x_n^*) be a sequence in B_{X^*} converging $\sigma(X^*, D)$ to zero. Since T is compact, we may assume there is an element h in $L_1(\lambda)$ to which (Tx_n^*) converges in norm and a.e. We need to show that $h = 0$ a.e. Let x^* be a weak* cluster point of the sequence (x_n^*). Then $x^*f = h$ a.e. But since (x_n^*) is converging $\sigma(X^*, D)$ to zero, $x^*|_D \equiv 0$. Hence, $0 = x^*f = h$ a.e.

Now assume T is weakly compact. Let (x_n^*) be a sequence in B_{X^*} converging $\sigma(X^*, D)$ to zero. By weak compactness of T, we may assume that (Tx_n^*) converges weakly to h. We want to show that $h = 0$ a.e.

Let $M_1 = \text{Cl}(\text{co}\{Tx_n^*\}_{n\geq 1})$, a closed and convex set containing h. There exists a sequence (y_n^*) in $\text{co}\{x_n^*\}_{n\geq 1}$ such that $y_n^* f \xrightarrow[n\to\infty]{} h$ a.e. Let z_1^* be a weak* cluster point of (y_n^*). Then $z_1^* \in w^*\text{-}\text{Cl}(\text{co}\{x_n^*\}_{n\geq 1})$ and $z_1^* f = h$ a.e. Let $M_2 = \text{Cl}(\text{co}\{Tx_n^*\}_{n\geq 2})$, a closed and convex set containing h. As before, we find an element $z_2^* \in w^*\text{-}\text{Cl}(\text{co}\{x_n^*\}_{n\geq 2})$ such that $z_2^* f = h$ a.e. Continuing this way we produce a sequence $(z_k^*) \in B_{X^*}$ such that

$$z_k^* \in w^*\text{-}\text{Cl}(\text{co}\{x_n^*\}_{n\geq k}) \subseteq w^*\text{-}\text{Cl}(\text{co}\{x_n^*\}_{n\geq k-1}) \quad \text{for all } k,$$

and

$$z_k^* f = h \quad \text{a.e. for all } k.$$

Let z^* be a weak* cluster point of (z_k^*). Then $z^* \in \bigcap_{k=1}^{\infty} w^*\text{-}\text{Cl}(\text{co}\{x_n^*\}_{n\geq k})$ and $z^* f = h$ a.e. If we can show that $z^*|_D \equiv 0$ then the proof is completed.

To obtain a contradiction, assume that there exists $x \in D$ such that $z^*(x) > \alpha > 0$. By passing to a subsequence, and after reindexing, we may assume that $z_k^*(x) > \alpha/2 > 0$ for all k. But $z_k^* \in w^*\text{-}\text{Cl}(\text{co}\{x_n^*\}_{n\geq k})$, so we can find an element, say $\sum_{i=1}^{l} \alpha_i x_{n_i}^*$, a convex combination, such that

$$\sum_{i=1}^{l} \alpha_i x_{n_i}^*(x) > \frac{\alpha}{4} > 0.$$

But this means that at least one of the $x_{n_i}^*(x)$'s must be larger than $\alpha/4$. Since we can do this for each k, we obtain a subsequence $(x_{n_k}^*)$ of (x_n^*) such that $x_{n_k}^*(x) > \alpha/4 > 0$ for all k, which contradicts the assertion that (x_n^*) is converging $\sigma(X^*, D)$ to zero. □

Since T maps bounded sequences which converge $\sigma(X^*, D)$ to zero, to sequences converging weakly to zero, the operator T_D is sequentially weak*-to-weak continuous and hence, weak*-to-weak continuous whenever D has Mazur's property. Hence, we have the following generalization of Corollary 4 of [9].

Theorem 2.5. *Let $f : \Omega \to X$ be Dunford integrable and T weakly compact. If f is determined by a subspace having Mazur's property, then f is Pettis integrable.*

A weakly measurable function $f : \Omega \to X$ is said to be *weakly bounded* if there is a constant $M > 0$ such that for each x^* in X^*,

$$|x^* f| \leq M \cdot \|x^*\|.$$

If X is a dual space, $X = Y$, then f is called weak* bounded if $|f(\cdot)y| \leq M \cdot \|y\|$ for all y in Y.

Lemma 2.6. *Assume $f : \Omega \to X$ is weakly measurable. There exists a countable partition π of Ω into measurable sets such that $f \cdot \chi_E$ is weakly bounded for all E in π.*

Consequently, there is a set F of arbitrarily small measure such that $f \cdot \chi_{\Omega|F}$ is weakly bounded.

Proof. For any $E \in \Sigma$ let $\Sigma^+(E) = \{F \subseteq E : F \in \Sigma \text{ and } \lambda(F) > 0\}$. For E_0 in $\Sigma^+(\Omega)$, fix an integer n and observe that one of the two mutually exclusive properties must hold

(i) There exists $F \in \Sigma^+(E_0)$ such that for all $x^* \in B^*$, $|(x^*f) \cdot \chi_F| < n$ a.e.

(ii) For each $E \in \Sigma^+(E_0)$, there exists $F \in \Sigma^+(E)$ and $x_F^* \in B^*$ such that $|(x_F^*f) \cdot \chi_F| \geq n$ and hence, $\|f(w)\| \geq n$ for all $w \in F$.

If (i) fails, a standard exhaustion argument shows that for all w in $E_0 \backslash K_n$, where K_n is of measure zero, $\|f(w)\| \geq n$. Consider the same two properties for the integer $n+1$. If (i) fails again, there exists a set K_{n+1} of measure zero such that $\|f(w)\| \geq n+1$ for all w in $E_0 \backslash K_{n+1}$. Continue through the integers one by one until reaching an integer N for which property (i) does not fail. Otherwise, if $K = \bigcup_{n=1}^{\infty} K_n$, a set of measure zero, then we see that for all w in $E_0 \backslash K$, $\|f(w)\| \geq n$ for all n, which clearly is impossible. Hence, each set of positive measure has a subset of positive measure on which f is weakly bounded. A standard exhaustion argument completes the proof. □

The above lemma allows us to write each weakly measurable function f in the form

$$f = \sum_{E \in \pi} f \cdot \chi_E ,$$

where π is a countable partition of Ω into measurable sets, and each $f \cdot \chi_E$ is weakly bounded. Since weakly bounded weakly measurable functions are Dunford integrable, this shows that any weakly measurable function f is "almost" Dunford integrable in the sense that, for any given $\varepsilon > 0$, there exists a measurable set E such that $\lambda(\Omega \backslash E) < \varepsilon$, and $f \cdot \chi_E$ is Dunford integrable.

Using Theorem 2.5 and Lemma 2.6 we prove the following:

Lemma 2.7. *Let $f : \Omega \to X$ be weakly measurable. Then f is determined by a subspace of X having Mazur's property if and only if f is determined by a WCG subspace of X.*

Proof. ($\Leftarrow$) Clear, since every WCG space has Mazur's property.

($\Rightarrow$) Assume f is determined by a subspace H of X having Mazur's property. Write $f = \sum_{n \geq 1} f \cdot \chi_{E_n}$, where $\{E_n : n = 1, 2, 3, \dots\}$ is a partition of Ω into measurable sets and $f \cdot \chi_{E_n}$ is weakly bounded, $n = 1, 2, 3, \dots$. Since f is determined by H, $f \cdot \chi_{E_n}$ is determined by H and hence, Pettis integrable by Theorem 2.5, $n = 1, 2, 3, \dots$. Thus, $f \cdot \chi_{E_n}$ is determined by a WCG subspace D_n of X, $n = 1, 2, 3, \dots$. Let K_n be a subset of B_X such that span(K_n) is dense in D_n, $n = 1, 2, 3, \dots$. If we let $K = \bigcup_{n \geq 1} (\frac{1}{n} K_n)$ then K is weakly compact and f is determined by the WCG subspace span(K) of X. □

Assume f is Pettis integrable. Then T is weakly compact (being weak*-to-weak continuous) and hence, the adjoint T^* is weakly compact. In particular, the set $\nu(\Sigma) = \{\nu(E) : E \in \Sigma\}$ is a relatively weakly compact subset of X. If we let D be the span of $\nu(\Sigma)$ then D is WCG. Furthermore, if $x^*|_D = 0$ then

$$0 = x^*(\nu E) = x^*(T^*(\chi_E)) = \int_E x^* f \, d\lambda ,$$

for all E in Σ. Consequently, $x^* f = 0$ a.e. Thus, Pettis integrable functions are determined by WCG subspaces. Together with Theorem 2.5 and Lemma 2.7 the above observation gives us the following characteristic of Pettis integrable functions:

Theorem 2.8. *Let $f : \Omega \to X$ be Dunford integrable. The following statements are equivalent:*

(a) *f is Pettis integrable.*
(b) *f is determined by a WCG space and T is weakly compact.*
(c) *f is determined by a space having Mazur's property and T is weakly compact.*

Proof. (a) $\Rightarrow$ (b) This is pointed out in the discussion following the proof of Lemma 2.7.

(b) $\Leftrightarrow$ (c) This is Lemma 2.7.

(c) $\Rightarrow$ (a) This is Theorem 2.5. □

Example II.3.3 of [5] shows that weak compactness of T cannot be omitted in the above theorem. In [5], Theorem II.3.7, it is shown that if a Banach space X does not have a copy of c_0, then any strongly measurable function into X is Pettis integrable whenever it is Dunford integrable, so the absence of c_0 replaces the requirement of T being weakly compact. Using Lemma 2.6 we can extend this theorem as follows:

Theorem 2.9. *Let D be a subspace of X, and assume D does not contain a copy of c_0. If D is WCG (has Mazur's property), then every Dunford integrable function f determined by D is Pettis integrable.*

Proof. Assume D is WCG. Since f is weakly measurable, there exists a countable partition π of Ω into measurable sets such that for each E in π, the function $f \cdot \chi_E$ is weakly bounded. Since f is determined by a WCG subspace, each $f \cdot \chi_E$ is determined by a WCG subspace, and hence, $f \cdot \chi_E$ is Pettis integrable for all E in π. This means that for each $F \in \Sigma$

$$(D)\text{-}\int_{F\cap E} f\,d\lambda = (P)\text{-}\int_{F\cap E} f\,d\lambda \in X.$$

For $x^* \in X^*$ and $F \in \Sigma$

$$\sum_{E\in\pi}\left|x^*\left((P)\text{-}\int_{F\cap E} f\,d\lambda\right)\right| = \sum_{E\in\pi}\left|\int_{F\cap E} x^* f\,d\lambda\right|$$
$$\leq \sum_{E\in\pi}\int_{F\cap E}|x^* f|\,d\lambda = \int_F |x^* f|\,d\lambda < \infty.$$

Since D has no copy of c_0, the Bessaga-Pelczynski characterization theorem, [4, p. 45], says that the series $\sum_{E\in\pi}(P)\text{-}\int_{F\cap E} f\,d\lambda$ is an unconditionally norm convergent series for all F in Σ.

Evidently $\sum_{E\in\pi}(P)\text{-}\int_{F\cap E} f\,d\lambda = (P)\text{-}\int_F f\,d\lambda$. □

An argument similar to the one used in the proof of Lemma 2.7 shows that if a function $f : \Omega \to X$ is the almost everywhere weak pointwise limit of a sequence (f_n) of Pettis integrable functions in the sense that

$$\text{for each } x^* \text{ in } X^*, \qquad x^* f = \lim_n x^* f_n \quad \text{a.e.},$$

then f is determined by a WCG subspace of X. Hence, if we know, or if we can show that f is Dunford integrable with countably additive indefinite integral (equivalently, the set $\{x^*f : x^* \in B_{X^*}\}$ is uniformly integrable) then f is Pettis integrable. In this way we can extend Theorem 3 of [8] to hold for nonperfect measure spaces.

Theorem 2.10. *Let $f : \Omega \to X$. If there is a sequence (f_n) of Pettis integrable functions from Ω to X such that*

(a) *The set $\{x^*f_n : x^* \in B_{X^*},\ n = 1, 2, 3, \dots\}$ is uniformly integrable, and*

(b) *for each x^* in X^*, $\lim x^*f_n = x^*f$ a.e.,*

then f is Pettis integrable and $\lim_n \int_E f_n\, d\lambda = \int_E f\, d\lambda$ weakly for each E in Σ.

Proof. As we have already pointed out, condition (b) implies that f is determined by a WCG subspace of X. It remains to show that f is Dunford integrable and the set $\{x^*f : x^* \in B_{X^*}\}$ is uniformly integrable, but that follows from Vitali's convergence theorem. □

Corollary 2.11. *Let $f : \Omega \to X$ be Dunford integrable, and assume X has no copy of c_0. The following statements are equivalent*:

(a) *f is Pettis integrable.*

(b) *There exists a sequence $f_n : \Omega \to X$ of Pettis integrable functions such that for each x^* in X^*, $x^*f = \lim_n x^*f_n$ a.e.*

Proof. (a) $\Rightarrow$ (b). Clearly

(b) $\Rightarrow$ (a). By Theorem 2.8, each f_n is determined by a WCG subspace of X. It follows that f is determined by a WCG subspace of X. An appeal to Theorem 2.9 concludes the proof. □

Remark. A slightly different version of Theorem 2.10 appears in [10], but the statement of that theorem is too general to be true as the following example shows.

Example. Let $\Omega = [0, 1]$ and $(\Omega, \Sigma, \lambda)$ be the Lebesgue measure space. For each n in $\mathbf{N}$ define the function

$$f_n : \Omega \to c_0, \qquad t \mapsto e_n\chi_\Omega,$$

where $\{e_n : n \in \mathbf{N}\}$ is the standard basis for c_0. Then

(i) For $x^* \in c_0^* = l_1$, $x^*f_n = 0$ a.e. implies $x^* \equiv 0$.

(ii) $\{x^*f_n : x^* \in B_{X^*},\ n = 1, 2, 3, \dots\} = \{\alpha\chi_\Omega : -1 \le \alpha \le 1\}$ is uniformly integrable. By theorem of [10], any weakly measurable function into c_0 would be Pettis integrable which we know is not true.

3. Functions determined by separable spaces

Let $f : \Omega \to X$ be weakly measurable and assume there exists a sequence (φ_n) of $X(X^{**})$ valued simple functions such that

$$(*) \qquad \text{for each } x^* \text{ in } X^*, \qquad x^*f = \lim_n x^*\varphi_n \quad \text{a.e.}$$

Write $\varphi_n = \sum_{k=1}^{m_n} x_{n_k}\chi_{E_{n_k}} (= \sum_{k=1}^{m_n} x_{n_k}^{**}\chi_{E_{n_k}})$. Let Σ_n be the algebra generated by $\{E_{n_k}\}_{k=1}^{m_n}$, and let $\sigma(\bigcup_{n=1}^{\infty} \Sigma_n) = \Sigma_\infty$ denote the complete σ-algebra generated by the collection (Σ_n). It is clear that x^*f is Σ_∞-measurable for all x^* in X^*, so if f is Dunford integrable, the set $\{x^*f : x^* \in X^*\} \subseteq L_1(\lambda, \Sigma_\infty) \subseteq L_1(\lambda)$ is separable. Hence, whenever a Dunford integrable function $f : \Omega \to X$ is a weak (weak*) a.e. pointwise limit of a sequence (φ_n) of simple functions in the sense of $(*)$ the range of the operator $T : X^* \to L_1(\lambda)$, $x^* \mapsto x^*f$ is separable. Lemma 3.1 shows that the converse is true. To prepare for the proof we introduce some notation.

If π is any finite partition of Ω into measurable sets and $\Sigma_\pi = \sigma(\pi)$ denotes the σ-algebra determined by π, the operator $E_\pi : L_1(\lambda) \to L_1(\lambda)$ defined by

$$E_\pi(g) = \sum_{E \in \pi} \left\{ \frac{1}{\lambda(E)} \int_E g \, d\lambda \right\} \chi_E ,$$

maps each element of $L_1(\lambda)$ onto its conditional expectation relative to Σ_π. If (π_n) is an increasing sequence of finite partitions of Ω into measurable sets, and $\Sigma_0 = \sigma(\bigcup_n \Sigma_{\pi_n})$ then, for each g in $L_1(\lambda)$, $E_{\pi_n}(g) \to E_{\Sigma_0}(g)$ a.e. and in $L_1(\lambda)$-norm, where $E_{\Sigma_0} : L_1(\lambda) \to L_1(\lambda)$ is the conditional expectation operator relative to Σ_0. In particular, if $(\Omega, \Sigma, \lambda)$ is a separable measure space, $E_{\pi_n}(g) \to g$ a.e. and in $L_1(\lambda)$-norm.

Lemma 3.1. *Let $f : \Omega \to X$ be Dunford integrable. Let $T : X^* \to L_1(\lambda)$, $x^* \mapsto x^*f$. The range of T is a separable subspace of $L_1(\lambda)$ if and only if there exists a sequence (φ_n) of X^{**} valued simple functions such that for all x^* in X^**

$$x^*f = \lim_n \varphi_n x^* \quad \text{a.e. and in } L_1(\lambda)\text{-norm}.$$

If f is Pettis integrable, the sequence (φ_n) can be chosen to be X-valued, and hence, f is determined by a separable subspace of X.

Proof. The sufficiency has been established. To prove the necessity, choose a countable subset A of X^* so that TA is dense in TX^*. There exists a countable collection (F_n) of sets in Σ such that if Σ_0 denotes the completion of the σ-algebra generated by (F_n), then Tx^* is Σ_0-measurable for all x^* in A.

For $n = 1, 2, 3, \ldots$, let Σ_n denote the finite algebra generated by $(F_i)_{i=1}^n$, let π_n be the atoms of Σ_n, and let Σ_∞ be the completion of the algebra $\bigcup \Sigma_n$. Then $\Sigma_\infty = \Sigma_0$.

Let E_n be the conditional expectation operator on $L_1(\lambda, \Sigma)$ relative to Σ_n. Then $E_n(x^*f) \to x^*f$ a.e. and in norm for all x^* in A. Fix any x^* in X^* and choose a sequence (x_n^*) in A such that $\|x_i^*f - x^*f\|_1 < 1/i$. Then

$$\begin{aligned}\|E_n(x^*f) - x^*f\|_1 &\le \|E_n(x^*f - x_i^*f)\|_1 + \|E_n(x_i^*f) - x_i^*f\|_1 + \|x_i^*f - x^*f\|_1 \\ &\le 2\|x^*f - x_i^*f\|_1 + \|E_n(x_i^*f) - x_i^*f\|_1 .\end{aligned}$$

Now if $\varepsilon > 0$ is given, choose i such that $1/i < \varepsilon/3$, and for that particular i choose N such that $\|E_n(x_i^*f) - x_i^*f\|_1 < \varepsilon/3$ for all $n \ge N$. Then,

$$\|E_n(x^*f) - x^*f\|_1 < 2\varepsilon/3 + \varepsilon/3 = \varepsilon ,$$

for all $n \ge N$. Note that for any x^* in X^*

$$E_n(x^*f) = \sum_{E \in \pi_n} \frac{1}{\lambda(E)} \left\{ \int_E x^*f \, d\lambda \right\} \chi_E = \left(\sum_{E \in \pi_n} \frac{1}{\lambda(E)} \left\{ (D)\text{-}\int_E f \, d\lambda \right\} \chi_E \right) (x^*),$$

so if we let $\varphi_n = \sum_{E\in\pi_n} \frac{1}{\lambda(E)} \cdot \{(D)\text{-}\int_E f\,d\lambda\}\chi_E$, then $\varphi_n x^* \to x^* f$ a.e. and in $L_1(\lambda)$-norm.

If f is Pettis integrable, then $(D)\text{-}\int_E f\,d\lambda = (P)\text{-}\int_E f\,d\lambda$ is in X for all E in Σ, so our simple functions are X-valued. Furthermore, if D is the closed linear span of $\bigcup \varphi_n(\Omega)$, then D is separable. If $x^*|_D = 0$, then $x^*\varphi_n = 0$ for all n and hence, $x^* f = \lim_n x^*\varphi_n = 0$ a.e. But this means that f is determined by D. □

We are now in a position to characterize Pettis integrable functions determined by separable spaces.

Theorem 3.2. *Let $f : \Omega \to X$ be Pettis integrable. The following statements are equivalent:*

(a) *f is determined by a separable subspace of X.*

(b) *There exists a sequence (φ_n) of X-valued simple functions such that for all $x^* \in X^*$*

$$x^* f = \lim_n x^* \varphi_n \quad \text{a.e. and in } L_1(\lambda)\text{-norm.}$$

(c) *T has a separable range.*

(d) *T^* has a separable range.*

Proof. (a) ⇒ (c) Assume f is determined by a separable subspace D of X. Then X^* has a countable $\sigma(X^*, D)$ dense subset. But T is $\sigma(X^*, D)$-to-weak continuous. Hence, T has a separable range.

(c) ⇒ (b) This is Lemma 3.1.

(b) ⇒ (d) Let $Z = \text{span}(\bigcup_n \varphi_n(\Omega))$. Then Z is separable. If $x^*|_Z = 0$ then $x^*\varphi_n = 0$ for all n and hence, $x^* f = 0$ a.e. We want to show that for any E in Σ, $T^*\chi_E$ is in Z. To obtain a contradiction, assume there exists a set E in Σ such that $T^*\chi_E$ is in $X\backslash Z$. Then there exists x^* in X^*, with $x^*|_Z = 0$, and such that $x^*(T^*\chi_E) = \int_E x^* f\,d\lambda > 0$. But

$$\int_E x^* f\,d\lambda = \lim_n \int_E x^*\varphi_n\,d\lambda = \lim_n x^*\left((\text{Bochner})\text{-}\int_E \varphi_n\,d\lambda\right),$$

and $(\text{Bochner})\text{-}\int_E \varphi_n\,d\lambda \in Z$, for all n.

(d) ⇒ (a) Let $\text{Range}(T^*) = D \subseteq X$. If $x^*|_D = 0$ then for all $E \in \Sigma$

$$x^*(T^*\chi_E) = \int_E x^* f\,d\lambda = 0.$$

Since this equation holds for all measurable E, we conclude that $x^* f = 0$ a.e. It follows that f is determined by D. But D is separable. □

Two weakly measurable functions $f : \Omega \to X$ and $g : \Omega \to X$ are said to be *weakly equivalent* if for each x^* in X^*,

$$x^* f = x^* g \quad \text{a.e.}$$

If X is a dual space, $X = Y^*$, then f and g are *weak equivalent* if $f(\cdot)y = g(\cdot)y$ a.e. for all y in Y.

Lemma 3.3. *For a strongly measurable function $g : \Omega \to X$, the following are equivalent:*

(1) *g is essentially bounded.*

(2) *g is weakly bounded.*

If X is a dual space, $X = Y^*$, these are equivalent to

(3) *There exists M such that $|g(w)(y)| \le M \cdot \|y\|$ a.e. for all y in Y.*

Proof. Suppose (2) (or (3)) holds. We prove (1). It suffices to show that if there exists a set E of positive measure such that $\|g(w)\| > M$ for all $w \in E$, then there exists $x_0^* \in B^*$ and a set G of positive measure such that $|x_0^* g(w)| \ge M$ for all $w \in G$.

By redefining g on a set of measure zero we may assume that g is a uniform limit of a sequence (φ_n) of countably-valued functions. Assume there exists a set E_0 of positive measure such that $\|g(w)\| > M$ for all $w \in E_0$. By restricting g to a subset of E_0, we may assume there exists an $\varepsilon > 0$ such that $\|g(w)\| > M + \varepsilon$ for all $w \in E_0$.

Choose $n \in N$ such that $\|g - \varphi_n\| < \varepsilon/4$. If $\varphi_n = \sum_{i=1}^{\infty} x_{ni}\chi_{E_{ni}}$, then there is an integer j such that the set $G = E_0 \cap E_j$ has a positive measure. Hence $\|g(w) - x_{nj}\| < \varepsilon/4$ for all $w \in G$. But $\|g(w)\| > M + \varepsilon$ for all $w \in G$, so $\|x_{nj}\| > M + 3\varepsilon/4$. Find $x_0^* \in B^*$ such that $x_0^*(x_{nj}) > \|x_{nj}\| - \varepsilon/4$. Then for all $w \in G$,

$$|x_0^* g(w)| \ge \big|\,|x_0^*(g(w) - x_{nj})| - |x_0^*(x_{nj})|\,\big| > M + \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = M + \frac{\varepsilon}{4}.$$

If $X = Y^*$ the element x_0^* can clearly be chosen to belong to Y. □

Let us assume we are given a weakly measurable function $f : \Omega \to X$ and we want to know if f is weakly (weak*) equivalent to a strongly measurable function $g : \Omega \to X(X^{**})$. By Lemma 2.6, we can assume that f is weakly bounded. Thus, by Lemma 3.3, g is essentially bounded and Bochner integrable [5]. Since f is weakly bounded its indefinite integral is countably additive and of bounded variation. Hence, to say that f is weakly (weak*) equivalent to a strongly measurable function is the same as saying that the indefinite integral of f is given by a Bochner integrable function. Indeed, if f is weakly equivalent to a strongly measurable function $g : \Omega \to X$ then f is Pettis integrable and for any E in Σ,

$$x^*\left((P)\text{-}\int_E f\,d\lambda\right) = \int_E x^* f\,d\lambda = \int_E x^* g\,d\lambda = x^*\left((B)\text{-}\int_E g\,d\lambda\right).$$

Since this equation holds for all x^* in X^*, we have

$$(P)\text{-}\int_E f\,d\lambda = (B)\text{-}\int_E g\,d\lambda.$$

Thus, if $\nu : \Sigma \to X$, $\nu(E) = (P)\text{-}\int_E f\,d\lambda$ is the indefinite integral of f, then it must be given by a Bochner integrable function, namely g.

Conversely, if the indefinite integral of f is given by a Bochner integrable function $g : \Omega \to X$, then for any E in Σ,

$$\int_E x^* f\,d\lambda = x^*\left((P)\text{-}\int_E f\,d\lambda\right) = x^*\left((B)\text{-}\int_E g\,d\lambda\right) = \int_E x^* g\,d\lambda.$$

Thus, $x^* f = x^* g$ a.e., and f is weakly equivalent to a strongly measurable function.

If f is not Pettis integrable, its indefinite integral, ν, has its range in X^{**}. As before, we have that f is weak* equivalent to a strongly measurable function if and only if ν is given by a Bochner integrable function $g : \Omega \to X^{**}$.

Proposition 3.4. *A Dunford integrable function* $f:\Omega\to X$ *is weak* equivalent to a strongly measurable function* $g:\Omega\to X^{**}$ *if and only if for each set* E *of positive measure there is a set* $F\subseteq E$ *of positive measure such that the operator* $T^*_{\chi_F}: L_\infty(\lambda)\to X^{**}$, $h\mapsto \int_F hf\,d\lambda$ *has a Bochner representable extension to* $L_1(\lambda)$.

Proof. Assume that f is weak* equivalent to a strongly measurable function $g:\Omega\to X^{**}$ and let $E\in\Sigma$ be of positive measure. By Lemma 2.6 there exists a set $F\subseteq E$ of positive measure such that f restricted to F is weakly bounded, say $|(x^*f)\chi_F|\le M\|x^*\|$ a.e. for some integer M. By Lemma 3.3 this implies that g restricted to F is essentially bounded. Hence, $g\chi_F$ is Bochner integrable.

For any $E\in\Sigma$ and any $x^*\in X^*$

$$(D)\text{-}\int_{E\cap F} f\,d\lambda(x^*)=\int_{E\cap F} x^*f\,d\lambda=\int_{E\cap F} x^*g\,d\lambda=(\text{Bochner})\text{-}\int_{E\cap F} g\,d\lambda(x^*).$$

Since this equation holds for all $x^*\in X^*$ we must have that

$$(D)\text{-}\int_{E\cap F} f\,d\lambda=(\text{Bochner})\text{-}\int_{E\cap F} g\,d\lambda \quad \text{for all } E\in\Sigma.$$

It follows that for any simple function $\varphi\in L_1(\lambda)$

$$T^*_{\chi_F}\varphi=\int_F \varphi f\,d\lambda=\int_F\varphi g\,d\lambda.$$

To see that $T^*_{\chi_F}$ is Bochner representable fix any $h\in L_1(\lambda)$. The function $hg:\Omega\to X$ is strongly measurable and $\int_F\|gh\|\,d\lambda\le M\|h\|_1$, so $hg\chi_F$ is Bochner integrable. Choose a sequence (φ_n) of simple functions such that $\varphi_n \xrightarrow[n\to\infty]{} h$ in $L_1(\lambda)$. Then

$$T^*_{\chi_F}h=\lim_n T^*_{\chi_F}\varphi_n=\lim_n\int_F\varphi_n g\,d\lambda.$$

But $\|\int_F hg\,d\lambda-\int_F\varphi_n g\,d\lambda\|=\|\int_F(h-\varphi_n)g\,d\lambda\|\le M\|h-\varphi_n\|_1\to 0$. Hence

$$T^*_{\chi_F}h=\lim_n\int_F\varphi_n g\,d\lambda=\int_F hg\,d\lambda.$$

Conversely, suppose that for every set E of positive measure there is a set $F\subseteq E$ of positive measure and that the operator $T^*_{\chi_F}$ has a Bochner representable extension to $L_1(\lambda)$. Fix an element $x^*\in X^*$ and a set $E\in\Sigma$. Then

$$\int_{F\cap E} x^*f\,d\lambda=T^*_{\chi_F}(\chi_E(x^*))=(\text{Bochner})\text{-}\int_{F\cap E} g\,d\lambda(x^*)=\int_{F\cap E}x^*g\,d\lambda.$$

Since this equation holds for all $E\in\Sigma$ $x^*f=x^*g$ almost everywhere on F. But x^* was arbitrary. Hence, $f\chi_F$ and $g\chi_F$ are weak* equivalent. A standard exhaustion argument provides us with a sequence (g_n, F_n) where the F_n's are disjoint sets of positive measure such that $\lambda(\Omega)=\lambda(\bigcup_{n=1}^\infty F_n)$ and the g_n's are such that $g_n\chi_{F_n}$ is weak* equivalent to $f\chi_{F_n}$ for all n. Without loss of generality we can assume that each g_n is zero outside F_n. Now if we define $g(w)=\sum_{n=1}^\infty g_n(w)$ if $w\in\bigcup_{n=1}^\infty F_n$ and zero otherwise then it is clear that g is strongly measurable and weak* equivalent to f. □

In view of the above proposition it is clear that if $f : \Omega \to X$ is weakly measurable and determined by a subspace D of X then:

(i) If D^{**} has the Radon-Nikodym Property [5], then all weakly measurable functions into X determined by D are weak* equivalent to strongly measurable functions into X^{**}, and those that are Pettis integrable are weakly equivalent to strongly measurable functions into X.

(ii) If D has the Radon-Nikodym Property, all Pettis integrable functions determined by D are weakly equivalent to strongly measurable functions into X.

Proposition 3.5. *All weakly measurable functions determined by reflexive spaces are weakly equivalent to strongly measurable functions.*

Proof. Let X be a Banach space, let D be a reflexive subspace of X, and let f be a weakly measurable function into X determined by D. Without loss of generality, we may assume f is weakly bounded. The operator $T_D : D^* \to L_1(\lambda)$, $d^* \mapsto d^*_{\text{ext}} f$ is weak-to-weak continuous and, since D is reflexive, T_D is in fact weak*-to-weak continuous. Thus, by Corollary 2.3, f is Pettis integrable. Since reflexive spaces have the Radon-Nikodym Property [5, Corollary 4, p. 82] the proof is complete. □

Remark. If $f : \Omega \to X$ is Pettis integrable, the following statements are equivalent:

(1) f is weakly equivalent to a strongly measurable function.

(2) There exists a sequence (φ_n) of simple functions such that for all $x^* \in X^*$

$$x^* \varphi_n \xrightarrow[n\to\infty]{} x^* f \quad \text{a.e.},$$

and $(\varphi_n(w))$ is relatively weakly compact a.e.

(3) There exists a sequence (φ_n) of simple functions such that for all $x^* \in X^*$

$$x^* \varphi_n \xrightarrow[n\to\infty]{} x^* f \quad \text{a.e.},$$

and for each set E of positive measure, there exists a set $F \subseteq E$ of positive measure such that $\bigcup_{n=1}^{\infty} \varphi_n(F)$ is relatively weakly compact.

Proof. (1) $\Rightarrow$ (3) Assume f is weakly equivalent to a strongly measurable function g. There exists a sequence (φ_n) of simple functions such that $\lim_n \|g - \varphi_n\| = 0$ a.e. Let E be any set of positive measure. By Egoroff's theorem, there exists a set $F \subseteq E$ of positive measure such that (φ_n) converges to g uniformly on F. Since the φ_n's are simple, the set $\bigcup_{n=1}^{\infty} \varphi_n(F)$ is totally bounded and hence, relatively weakly compact.

(3) $\Rightarrow$ (2) If not, there exists a set E of positive measure such that for each $w \in E$, the sequence $(\varphi_n(w))$ has no weakly convergent subsequence, contradicting (3).

(2) $\Rightarrow$ (1) Assuming (2), find a set E of measure zero such that off E the set $\{\varphi_n(w)\}$ is relatively weakly compact. For each $w \notin E$, choose a weak cluster point x_w of $(\varphi_n(w))$ and define a function $g : \Omega \to X$ by the equation

$$g(w) = \begin{cases} 0, & \text{if } w \in E, \\ x_w, & \text{if } w \notin E. \end{cases}$$

Then g is separably valued. By definition, g is weakly equivalent to f and hence, weakly measurable. Being separably valued, g is strongly measurable.

4. Integration in dual spaces

Lemma 4.1. *Let $f : \Omega \to X^*$ be weak* measurable.*

(a) *There exists a countable partition π of Ω into measurable sets such that for each E in π, $f \cdot \chi_E$ is weak* bounded.*

(b) *If f is weak* integrable and the set $\{f(\cdot)x : \|x\| \leq 1\}$ is separable, then there exists a sequence (π_n) of finite partitions of Ω into measurable sets such that*

(i) *π_{n+1} refines π_n, i.e. each member of π_{n+1} is contained in a member of π_n,*

(ii) *if we let $\varphi_n = \sum_{E\in\pi_n}\{\frac{1}{\lambda(E)}(w^*\text{-}\int_E f\,d\lambda)\}\chi_E$, then for all $x \in X$,*

$$f(\cdot)x = \lim_{n\to\infty} \varphi_n(\cdot)x \quad a.e.$$

Proof. (a) This is essentially Lemma 2.6.

(b) There exists a countable set $A \subseteq \{x : \|x\| \leq 1\}$ such that $\{f(\cdot)x : x \in A\}$ is norm dense in $\{f(\cdot)x : \|x\| \leq 1\}$. Since A is countable, there exists an increasing sequence (π_n) of finite partitions of Ω into measurable sets such that if Σ_n denotes the (trivial) σ-algebra generated by π_n and $\Sigma_\infty = \sigma(\bigcup \Sigma_n)$, then $f(\cdot)x$ is Σ_∞-measurable for all x in A. Let E_n denote the conditional expectation operator on $L_1(\lambda)$ with respect to Σ_n and let E_∞ denote the conditional expectation operator on $L_1(\lambda)$ with respect to Σ_∞. Then $E_\infty(f(\cdot)x) = f(\cdot)x$ for all $x \in A$. Hence, $E_n(f(\cdot)x)$ converges a.e. and in L_1-norm to $f(\cdot)x$ for all x in A.

Let x be any element in $\{x : \|x\| \leq 1\}$. Find a sequence (x_n) in A such that

$$\|f(\cdot)x - f(\cdot)x_n\|_1 \xrightarrow[n\to\infty]{} 0.$$

Then

$$\begin{aligned}\|E_\infty(f(\cdot)x) - f(\cdot)x\|_1 &\leq \|E_\infty(f(\cdot)x) - E_\infty(f(\cdot)x_n)\|_1 + \|E_\infty(f(\cdot)x_n) - f(\cdot)x\|_1 \\ &\leq \|E_\infty\| \cdot \|f(\cdot)x - f(\cdot)x_n\|_1 + \|f(\cdot)x - f(\cdot)x_n\|_1.\end{aligned}$$

Hence, $E_\infty(f(\cdot)x) = f(\cdot)x$ a.e. for all $x \in \{x : \|x\| \leq 1\}$. Then $E_n(f(\cdot)x)$ converges almost everywhere and in L_1-norm to $f(\cdot)x$ for all x in $\{x : \|x\| \leq 1\}$.

But $E_n(f(\cdot)x) = \sum_{E\in\pi_n}\{\frac{1}{\lambda(E)}(w^*\text{-}\int_E f\,d\lambda)\}\cdot\chi_E(\cdot)(x)$ for all n and all x in X, so if we define for each n a simple function

$$\varphi_n = \sum_{E\in\pi_n}\left\{\frac{1}{\lambda(E)}\left(w^*\text{-}\int_E f\,d\lambda\right)\right\}\cdot\chi_E,$$

then for all x in X, $f(\cdot)x = \lim_{n\to\infty}\varphi_n(\cdot)x$ a.e. □

Following [2] we define the *weak*-core of f over E*, denoted by $\operatorname{cor}_f^*(E)$, to be that subset of X^* given by

$$\operatorname{cor}_f^*(E) = \bigcap_{\lambda(A)=0} w^*\text{-}\operatorname{clco} f(E\backslash A).$$

Lemma 4.2 [2]. *If $f : \Omega \to X^*$ is weak* integrable then, for each $E \in \Sigma$,*

$$\operatorname{cor}_f^*(E) = w^*\text{-}\operatorname{clco}\left\{ \frac{1}{\lambda(F)} \left(w^*\text{-}\int_F f\, d\lambda \right) : F \subseteq E\,,\ F \in \Sigma\,,\ \lambda(F) > 0 \right\}.$$

Proof (R. Geitz). Let F be a subset of E of positive measure and let A be a set of measure zero. If $(1/\lambda(F))(w^*\text{-}\int_F f\,d\lambda)$ is not in $w^*\text{-}\operatorname{clco} f(F\backslash A)$ there exists an element x in X such that

$$\frac{1}{\lambda(F)} \int_F f(\cdot)x\, d\lambda > \sup\{f(w)x : w \in F\backslash A\}.$$

By integrating over $F\backslash A$ we get

$$\int_F f(\cdot)x\, d\lambda > \int_F f(\cdot)x\, d\lambda,$$

which is a contradiction. Hence, $(1/\lambda(F))(w^*\text{-}\int_F f\,d\lambda) \in w^*\text{-}\operatorname{clco} f(F\backslash A) \subseteq w^*\text{-}\operatorname{clco} f(E\backslash A)$. It follows that $w^*\text{-}\operatorname{clco}\{(1/\lambda(F))(w^*\text{-}\int_F f\,d\lambda) : F \subseteq E\,,\ F \in \Sigma\,,\ \lambda(F) > 0\} \subseteq \operatorname{cor}_f^*(E)$.

To prove the opposite inclusion, let x^* be in $\operatorname{cor}_f^*(E)$. It suffices to show that for any x in the unit ball of X,

$$x^*(x) \geq \inf\left\{ \frac{1}{\lambda(F)} \int_F f(\cdot)x\, d\lambda : F \subseteq E\,,\ F \in \Sigma\,,\ \lambda(F) > 0 \right\}.$$

To that end, fix $\varepsilon > 0$. Find a countable partition π of Ω into measurable sets and a sequence $(C_E)_{E\in\pi}$ such that for any E in π

$$|f(w)x - C_E| \leq \frac{\varepsilon}{4} \quad \text{for all } w \text{ in } E.$$

Note that if E is any set in π of positive measure and w is in E, then

$$\left| f(w)x - \frac{1}{\lambda(E)} \int_E f(\cdot)x\, d\lambda \right| < \frac{\varepsilon}{2}.$$

Let A be the union of all zero sets of π. Then x^* is in $w^*\text{-}\operatorname{clco} f(E\backslash A)$, so there exists a finite convex combination $\sum \alpha_i f(w_i)$ such that w_i is in $E\backslash A$ and $\|x^* - \sum \alpha_i f(w_i)\| < \varepsilon/2$. Thus, we have

$$\left| x^*(x) - \sum \alpha_i f(w_i)(x) \right| < \frac{\varepsilon}{2}.$$

If E_i is the element of π that contains w_i, then

$$\left| x^*(x) - \sum \alpha_i \frac{1}{\lambda(E_i)} \int_{E_i} f(\cdot)x\, d\lambda \right| < \varepsilon.$$

Since ε was arbitrary, it follows that

$$x^*(x) \geq \inf\left\{ \frac{1}{\lambda(F)} \int_F f(\cdot)x\, d\lambda : F \subseteq E\,,\ F \in \Sigma\,,\ \lambda(F) > 0 \right\}. \quad \square$$

Lemma 4.3. *Let $f : \Omega \to X^*$ be a weak* integrable function and assume that the set $\{f(\cdot)x : \|x\| \leq 1\} \subseteq L_1(\lambda)$ is separable. Then*

(a) *$\operatorname{cor}_f^*(\Omega)$ is a weak* separable subset of X^*.*

(b) *f is weak* equivalent to a weak* measurable function that takes its range in* $\operatorname{cor}^*_f(\Omega)$.

Proof. Since $\{f(\cdot)x : \|x\| \le 1\}$ is separable, Lemma 4.1(b) provides us with a sequence (φ_n) such that each φ_n takes its range in $\operatorname{cor}^*_f(\Omega)$ and for all $x \in X$, $\varphi_n(\cdot)x$ converges almost everywhere to $f(\cdot)x$.

To prove (a), observe that if F is any subset of Ω of positive measure, then for each $x \in X$,

$$\left(w^*\text{-}\int_F f\,d\lambda\right)(x) = \int_F f(\cdot)x\,d\lambda = \lim_{n\to\infty}\int_F \varphi_n(\cdot)x\,d\lambda$$

$$= \lim_{n\to\infty}\sum_{E\in\pi_n}\frac{1}{\lambda(E)}\left(w^*\text{-}\int_E f\,d\lambda\right)\lambda(F\cap E)(x).$$

$$\therefore \frac{1}{\lambda(F)}\left(w^*\text{-}\int_F f\,d\lambda\right)(x) = \lim_{n\to\infty}\sum_{E\in\pi_n}\frac{\lambda(E\cap F)}{\lambda(F)}\left\{\frac{1}{\lambda(E)}\left(w^*\text{-}\int_E f\,d\lambda\right)\right\}(x).$$

$$\therefore \operatorname{cor}^*_f(\Omega) \subseteq w^*\operatorname{clco}\left(\bigcup_n \varphi_n(\Omega)\right) \subseteq \operatorname{cor}^*_f(\Omega).$$

To prove (b), we first assume that f is weak* bounded by 1, i.e., for each $x \in X$,

$$|f(\cdot)x| \le \|x\| \quad \text{a.e.}$$

Then for any set $E \in \Sigma$ with $\lambda(E) > 0$,

$$\left\|\frac{1}{\lambda(E)}\left(w^*\text{-}\int_E f\,d\lambda\right)\right\| \le 1.$$

Indeed, $\|\frac{1}{\lambda(E)}(w^*\text{-}\int_E f\,d\lambda)\| \le \sup\{\frac{1}{\lambda(E)}\int_E |f(\cdot)x|\,d\lambda : \|x\| \le 1\} \le 1$. Hence, $\operatorname{cor}^*_f(\Omega)$ is bounded. Let g be any weak* cluster point of the sequence (φ_n). Then $g(\Omega) \subseteq \operatorname{cor}^*_f(\Omega)$ and g is weak* equivalent to f. In particular, g is weak* measurable. For the general case, observe that if E is a set of positive measure and $f\cdot\chi_E$ is weak* bounded, the $\operatorname{cor}^*_f(E)$ is bounded. Find a countable partition P of Ω such that $f\cdot\chi_E$ weak* bounded for each E in P. For each E in P let π_{nE} denote the common refinement of π_n and $\{\Omega\backslash E, E\}$. Then

$$\psi_n = \sum_{\substack{F\in\pi_{nE}\\ F\subseteq E}}\frac{1}{\lambda(F)}\left(w^*\text{-}\int_F f\,d\lambda\right)\cdot\chi_F,$$

is zero outside E, and otherwise takes its values in $\operatorname{cor}^*_f(E)$. If we let g_E be a weak* cluster point of (ψ_n), then g_E is weak* equivalent to $f\cdot\chi_E$. If we define a function g by the equation

$$g = \sum_{E\in P} g_E,$$

then g takes its range in $\operatorname{cor}^*_f(\Omega)$ and is weak* equivalent to f. □

In [3] Davis, Figiel, Johnson, and Pelczynski prove that for any WCG space X, there exists a reflexive space R and a one-to-one bounded linear operator $S : R \to X$ onto a dense linear subspace of X. Using this result we prove the following:

Lemma 4.4. *If X^* is a dual of a WCG space X and $f : \Omega \to X^*$ is weak* integrable, then $\{f(\cdot)x : \|x\| \leq 1\}$ is separable.*

In particular, $\mathrm{cor}^*_f(\Omega)$ *is weak* separable.*

Proof. There exists a reflexive space R and a one-to-one bounded linear operator $S : R \to X$ onto a dense linear subspace of X. Then $S^* : X^* \to R^*$ is one-to-one and is onto a dense linear subspace, since R is reflexive.

Let $f : \Omega \to X^*$ be weak* integrable and consider the function $S^*f : \Omega \to R^*$. It is clear that S^*f is weakly measurable and Dunford integrable. Hence, S^*f is Pettis integrable and weakly equivalent to a strongly measurable function $g : \Omega \to R^*$ by Proposition 3.4. Find a sequence (ψ_n) of R^*-valued simple functions such that

$$\|g - \psi_n\| \xrightarrow[n\to\infty]{} 0 \quad \text{a.e.}$$

Since S^* is onto a dense subspace of R^*, for each n we can find a simple function $f_n : \Omega \to X^*$ such that

$$\|S^*f_n - \psi_n\|_\infty = \sup_{w\in\Omega} \|S^*f_n(w) - \psi_n(w)\| < \frac{1}{2^n}.$$

Then we have $\|g - S^*f_n\| \leq \|g - \psi_n\| + \|\psi_n - S^*f_n\| \xrightarrow[n\to\infty]{} 0$ almost everywhere. But g is weakly equivalent to S^*f, so for all $r \in R$ $(= R^{**})$

$$\begin{aligned} r(S^*f) &= r(g) \quad \text{a.e.} \\ &= \lim_n r(S^*f_n) \quad \text{a.e.} \end{aligned}$$

Hence, for all $r \in R$,

$$(Sr)f = \lim_n (Sr)f_n \quad \text{a.e.}$$

Consider the operator $T_f : X \to L_1(\lambda)$, $x \mapsto f(\cdot)x$. If we let $T : R \to L_1(\lambda)$, $r \mapsto rS^*f$, then, for all $r \in R$

$$\begin{aligned} T(r) &= T_f(Sr) = T_fS(r). \\ \therefore T(R) &= T_fS(R). \end{aligned}$$

But $T(R)$ is separable, by Proposition 3.4 and Theorem 3.2, and $S(R)$ is dense in X. Hence,

$$T_f(X) = T_f(\mathrm{Cl}(SR)) \subseteq \mathrm{Cl}(T_fS(R)).$$

It follows that $\{f(\cdot)x : \|x\| \leq 1\}$ is separable. □

Using Lemmas 4.3 and 4.4 we characterize Pettis integrable functions into duals of WCG spaces.

Theorem 4.5. *If X^* is a dual of a WCG space X, and if $f : \Omega \to X^*$ is weakly measurable and weakly bounded, then the following are equivalent:*

(a) *f is Pettis integrable.*

(b) *f is determined by a separable subspace of X^*.*

(c) *There exists a sequence (φ_n) of simple functions such that for all $x^{**} \in X^{**}$,*

$$x^{**}f = \lim_{n\to\infty} x^{**}\varphi_n \quad a.e.$$

(d) *f is weakly equivalent to a Pettis integrable function $g : \Omega \to \mathrm{cor}^*_f(\Omega)$.*

Proof. (a) ⇒ (b) Assume we are given a Pettis integrable function $f : \Omega \to X^*$. Let $S : R \to X$ and $T_f : L_1(\lambda)$ be as in the proof of Lemma 4.4. If f is Pettis integrable, the operator $T : X^{**} \to L_1(\lambda)$ is weak*-to-weak continuous, and since X is weak* dense in X^{**},

$$TX^{**} = T(w^*\text{-Cl}(X)) \subseteq w\text{-Cl}(T(X)) = w\text{-Cl}(T_f(X)) \subseteq w\text{-Cl}(T_fS(R)).$$

But by Lemma 4.4, $T_fS(R)$ is separable. Hence, TX^{**} is separable and by Theorem 3.2, f is determined by a separable subspace.

(b) $\Rightarrow$ (c) This follows from Theorem 3.2.

(c) $\Rightarrow$ (d) By Theorem 3.2, f is Pettis integrable and $\{f(\cdot)x : \|x\| \le 1\}$ is separable. By Lemma 4.3(b), f is weak* equivalent to a weak* measurable function $g : \Omega \to \text{cor}^*_f(\Omega)$. To show that g is weakly measurable, we use the fact that g takes its range $\text{cor}^*_f(\Omega)$. By Lemma 4.3(a), $\text{cor}^*_f(\Omega)$ is a weak* separable subset of X^* and hence, generates a weak* separable subspace D of X^*. A theorem of Amir and Lindenstrauss [1] provides us with a projection $P : X \to X$ such that PX is separable, and $D \subseteq P^*X^*$. This means that we can write $X = X_1 \oplus X_2$ and $X^* = X_1^* \oplus X_2^*$ where X_2 is separable and $D \subseteq X_2^*$. Since X_2 is separable, there exists a set $E \in \Sigma$ of measure zero such that off E,

$$f(\cdot)x_2 = g(\cdot)x_2,$$

for all $x_2 \in X_2$. Hence, off E

$$(f(\cdot) - g(\cdot))x_2 = 0,$$

for all $x_2 \in X_2$. Consequently $x_2^{**}f = x_2^{**}g$ a.e. for all $x_2^{**} \in X_2^{**}$ and hence, g is weakly measurable as a function into X_2^*. Then g is weakly measurable into X^*. Since f is Pettis integrable, $(P)\text{-}\int_E f\,d\lambda = w^*\text{-}\int_E f\,d\lambda \in X_2^*$ for all $E \in \Sigma$. Therefore f is determined by X_2^*. If $x^{**} \in X^{**}$ write $x^{**} = x_1^{**} + x_2^{**}$ where $x_i^{**} \in X_i^{**}$, $i = 1, 2$. Then

$$\begin{aligned} x^{**}f &= (x_1^{**} + x_2^{**})f = (0 + x_2^{**})f \quad \text{a.e.} \\ &= (0 + x_2^{**})g \quad \text{a.e.} \\ &= (x_1^{**} + x_2^{**})g \\ &= x^{**}g. \end{aligned}$$

Hence, f is weakly equivalent to g. □

Corollary 4.6. *If X is isomorphic to a subspace of a dual of a WCG space and $f : \Omega \to X$ is a weakly bounded and weakly measurable function, then the following are equivalent*:

(a) *f is Pettis integrable.*

(b) *f is determined by a separable subspace of X.*

Proof. (b) $\Rightarrow$ (a) This is Theorem 2.5.

(a) $\Rightarrow$ (b) Let Z be a dual of a WCG space and $S : X \to S(X) \subseteq Z$ an isomorphism. If f is a Pettis integrable function into X, then Sf is a Pettis integrable function into Z and, by Theorem 4.6, Sf is determined by a separable subspace Y. By Theorem 3.2(d), the set $D = \text{span}\{(P)\text{-}\int_E Sf\,d\lambda : E \in \Sigma\}$ is contained in Y. But if $z^*|_D = 0$, then $\int_E z^*Sf\,d\lambda = 0$ for all $E \in \Sigma$, and consequently $z^*Sf = 0$ a.e. Hence, we can assume that Sf is determined by D. The space $S^{-1}(D)$ is a separable subspace of X and, since $(P)\text{-}\int_E Sf\,d\lambda = S((P)\text{-}\int_E f\,d\lambda)$ for all $E \in \Sigma$, the range of the indefinite integral of f is contained in $S^{-1}(D)$. Hence, f is determined by a separable subspace. □

Corollary 4.7. *If X has a WCG dual and $f : \Omega \to X$ is a weakly bounded weakly measurable function, then f is a Pettis integrable if and only if it is determined by a separable subspace of X.*

Proof. Note that $f : \Omega \to X$ is Pettis integrable if and only if it is Pettis integrable when viewed as a function into X^{**}. If X has a WCG dual then it is isomorphic to a subspace of a dual of a WCG space. □

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